

# UNIVERSAL PLANE CURVE AND MODULI SPACES OF 1-DIMENSIONAL COHERENT SHEAVES

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**ABSTRACT.** We show that the universal plane curve  $M$  of fixed degree  $d \geq 3$  can be seen as a closed subvariety in a certain Simpson moduli space of 1-dimensional sheaves on  $\mathbb{P}_2$  contained in the stable locus. The universal singular locus coincides with the subvariety of  $M$  consisting of sheaves that are not locally free on their support. It turns out that the blow up  $\mathrm{Bl}_{M'} M$  may be naturally seen as a compactification of  $M_B = M \setminus M'$  by vector bundles (on support).

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## 1. INTRODUCTION

**1.1. Motivation.** Simpson showed in [12] that for an arbitrary smooth projective variety  $X$  and for an arbitrary numerical polynomial  $P \in \mathbb{Q}[m]$  there is a coarse moduli space  $M := M_P(X)$  of semi-stable sheaves on  $X$  with Hilbert polynomial  $P$ , which turns out to be a projective variety.

In general  $M$  contains a closed subvariety of sheaves that are not locally free on their support. Its complement  $M_B$  is then an open dense subset whose points are sheaves that are locally free on their support. So, one could consider  $M$  as a compactification of  $M_B$ . We call the sheaves from the boundary  $M \setminus M_B$  *singular*. It is an interesting question whether and how one could replace the boundary of singular sheaves by one which consists entirely of vector bundles with varying and possibly reducible supports. This problem for one-dimensional sheaves on a projective plane was dealt with in [5] and [4]. The case of torsion-free sheaves on a surface is considered in [10].

It is known (see [6], [3]) that the universal plane cubic curve may be identified with the fine Simpson moduli space of stable coherent sheaves on  $\mathbb{P}_2$  with Hilbert polynomial  $3m + 1$ . In [4] it has been shown that the blowing-up of the universal plane cubic curve along its universal singular locus may be seen as a construction which substitutes the sheaves which are not vector bundles (on their 1-dimensional support) by vector bundles (on support).

**1.2. Main results of the paper.** The main aim of this note is to show the following:

- the universal plane curve of fixed degree  $d \geq 3$  can be seen as a closed subvariety of codimension  $\frac{d(d-3)}{2}$  in the Simpson moduli space of semistable sheaves on  $\mathbb{P}_2$  with Hilbert polynomial

$$dm + \frac{d(d-3)}{2} + 1$$

contained in the stable locus (Proposition 3.2);

- the blowing up along the universal singular locus can be seen as a process which substitutes the singular sheaves, i.e., those which are not locally free on their support, by vector bundles (on support) (Theorem 5.5).

This generalizes the construction presented in [5], [4]. Moreover, some important details omitted for the sake of brevity in [4] are presented (in bigger generality) here.

**1.3. Some notations and conventions.** In this paper we use notations and constructions from [4], in particular  $\mathbb{k}$  is an algebraically closed field of characteristic zero, we work in the category of separated schemes of finite type over  $\mathbb{k}$  and call them varieties, using only their closed points. We do not restrict ourselves to reduced or irreducible varieties. Dealing with homomorphism between direct sums of line bundles and identifying them with matrices, we consider the matrices acting on elements from the right, i. e., the composition  $X \xrightarrow{A} Y \xrightarrow{B} Z$  is given by the matrix  $A \cdot B$ .

In [4] surfaces  $D(p)$  were defined for every point  $p \in \mathbb{P}_2$ .  $D(p)$  consists of two irreducible components  $D_0(p)$  and  $D_1(p)$ ,  $D_0(p)$  being the blow up of  $\mathbb{P}_2$  at  $p$  and  $D_1(p)$  being another projective plane, such that these components intersect along the line  $L(p)$  which is the exceptional divisor of  $D_0(p)$ . Each surface  $D(p)$  can be defined as the subvariety in  $\mathbb{P}_2 \times \mathbb{P}_2$  with equations  $u_0x_1, u_0x_2, u_1x_2 - u_2x_1$  where the  $x_i$  respectively  $u_i$  are the homogeneous coordinates of the first respectively second  $\mathbb{P}_2$ , such that the first projection contracts  $D_1(p)$  to  $p$  and describes  $D_0(p)$  as the blow up. As in [4],  $\mathcal{O}_{D(p)}(a, b)$  denotes the invertible sheaf induced by  $\mathcal{O}_{\mathbb{P}_2}(a) \boxtimes \mathcal{O}_{\mathbb{P}_2}(b)$ .

**1.4. Structure of the paper.** In Section 2 we describe the universal curve as a quotient of a space of certain injective morphisms between rank 2 vector bundles on  $\mathbb{P}_2$ . In Section 3 we show that the universal curve is a subvariety of an appropriate Simpson moduli space. Proposition 3.2 is proved here. In Section 4 we identify the universal singular locus with the subvariety of singular sheaves in  $M$ . In Section 5 we prove Theorem 5.5, i. e., we show that the blowing up along the universal singular locus can be seen as a process which substitutes the singular sheaves by vector bundles (on support).

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## 2. UNIVERSAL CURVE AS A QUOTIENT

Let  $V$  be a 3-dimensional vector space over  $\mathbb{k}$ . Let  $\mathbb{P}_2 = \mathbb{P}V$  be the corresponding projective space. Let  $S^dV$  be the  $d$ -th symmetric power of  $V$ . Then  $\mathbb{P}_N = \mathbb{P}(S^dV)$  may be seen as the space of plane curves of degree  $d$ . Its dimension is  $N = \frac{(d+2)(d+1)}{2} - 1$ . Recall that a curve is identified with its equation up to multiplication by a non-zero constant. Assume that  $d \geq 3$ .

Consider the universal plane curve of degree  $d$

$$M = \{(C, p) \mid p \in C\} = \{(\langle f \rangle, \langle x \rangle) \in \mathbb{P}_N \times \mathbb{P}_2 \mid f(x) = 0\}.$$

This is a smooth projective variety of dimension  $N + 1 = \frac{(d+2)(d+1)}{2}$ .

Let  $X$  be the space of morphisms  $2\mathcal{O}_{\mathbb{P}_2}(-d+1) \xrightarrow{A} \mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2}$ ,  $A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$ , with linear independent  $z_1$  and  $z_2$  and with non-zero determinant. Note that we consider matrices acting on the right. Then one sees that  $X$  is an open subvariety in the affine variety  $\text{Hom}(2\mathcal{O}_{\mathbb{P}_2}(-d+1), \mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2})$ , which is isomorphic to  $\mathbb{k}^{d^2+d+6}$ .

We fix a basis  $\{x_0, x_1, x_2\}$  of  $H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$  and for  $A \in X$  we will write

$$\begin{aligned} z_1 &= a_0x_0 + a_1x_1 + a_2x_2, & z_2 &= b_0x_0 + b_1x_1 + b_2x_2, \\ q_1 &= \sum_{\substack{i,j \geq 0, \\ i+j \leq d-1}} A_{ij} x_0^{d-1-i-j} x_1^i x_2^j, & q_2 &= \sum_{\substack{i,j \geq 0, \\ i+j \leq d-1}} B_{ij} x_0^{d-1-i-j} x_1^i x_2^j. \end{aligned}$$

Since all morphism in  $X$  are injective,  $X$  may be seen as a parameter space of sheaves given by resolutions

$$(1) \quad 0 \rightarrow 2\mathcal{O}_{\mathbb{P}_2}(-d+1) \xrightarrow{\begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}} \mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2} \rightarrow \mathcal{F} \rightarrow 0.$$

**Remark 2.1.** *One can easily see that the Hilbert polynomial of such sheaves is  $dm + \frac{d(3-d)}{2} + 1$ .*

There is a morphism  $X \xrightarrow{\nu} M$ ,  $\begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} \mapsto (\langle z_1 q_2 - z_2 q_1 \rangle, z_1 \wedge z_2)$ , where  $z_1 \wedge z_2$  denotes the common zero of  $z_1$  and  $z_2$ .

**Lemma 2.2.**  $\nu$  is surjective.

*Proof.* Let  $(f, p) \in M$ . Choose two linear independent linear forms  $z_1$  and  $z_2$  such that  $p = z_1 \wedge z_2$ . Since  $f(p) = 0$ , one can write  $f = z_1 q_2 - z_2 q_1$  for some forms  $q_1$  and  $q_2$  of degree  $d-1$ . Then  $\begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$  is a preimage of  $(f, p)$ .  $\square$

**Lemma 2.3.** *Two matrices  $A_1, A_2 \in X$  lie in the same fibre of  $\nu$  if and only if there exist  $g \in \mathrm{GL}_2(\mathbb{k})$  and  $h = \begin{pmatrix} \lambda & q \\ 0 & \mu \end{pmatrix} \in \mathrm{Aut}(\mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2})$  such that  $gA_1h = A_2$ .*

*Proof.* It is clear that  $gA_1h = A_2$  implies that both  $A_1$  and  $A_2$  lie in the same fibre of  $\nu$ . Let us assume that  $A_1$  and  $A_2$  lie in the same fibre of  $\nu$ . Then in particular  $p(A_1) = p(A_2)$  and multiplying by an appropriate  $g \in \mathrm{GL}_2(\mathbb{k})$  we may assume that  $A_1 = \begin{pmatrix} z_1 & q'_1 \\ z_2 & q'_2 \end{pmatrix}$  and  $A_2 = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$ . Since  $\langle z_1 q_2 - z_2 q_1 \rangle = \langle z_1 q'_2 - z_2 q'_1 \rangle$  we obtain  $z_1 q_2 - z_2 q_1 = \xi(z_1 q'_2 - z_2 q'_1)$  for some  $\xi \in \mathbb{k}^*$  and thus multiplying the second column of  $A_1$  by  $\xi$  we may assume that  $\xi = 1$ . Then  $z_1(q_2 - q'_2) - z_2(q_1 - q'_1) = 0$  and hence  $\begin{pmatrix} q_1 - q'_1 \\ q_2 - q'_2 \end{pmatrix} = q \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$  for some form  $q$  of degree  $d-1$ . In other words

$$\begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} = \begin{pmatrix} z_1 & q'_1 \\ z_2 & q'_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}.$$

This proves the lemma.  $\square$

Note that the group  $G = \mathrm{Aut}(2\mathcal{O}_{\mathbb{P}_2}(-d+1)) \times \mathrm{Aut}(\mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2})$  naturally acts on  $X$  by the rule  $(g, h) \cdot A = gAh^{-1}$ . The last two lemmas show that  $M$  is an orbit space of this action.

**Lemma 2.4.** *The stabilizer of an arbitrary element in  $X$  coincides with the group*

$$St = \{((\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix})), ((\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix})) \mid \lambda \in \mathbb{k}^*\}.$$

*Proof.* Let  $A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix}$  and suppose  $g$  and  $h$  satisfy  $gA = Ah$ . Write  $h = \begin{pmatrix} \lambda & q \\ 0 & \mu \end{pmatrix}$ . Then  $g\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$  and  $(g - \lambda \cdot \mathrm{id})\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$ . Since  $z_1$  and  $z_2$  are linear independent, one concludes that  $g = \lambda \cdot \mathrm{id}$ . Then  $A = \lambda^{-1}Ah$  and since  $\det A \neq 0$  one concludes that  $\lambda = \mu$  and  $z_1 q = z_2 q = 0$ . Hence  $h = \lambda \cdot \mathrm{id}$ . This completes the proof.  $\square$

Denote the group  $G/St$  by  $\mathbb{P}G$ . Then  $\mathbb{P}G$  acts freely on  $X$ .

**Lemma 2.5.** *There is a local section of  $\nu$ .*

*Proof.* It is enough to prove the existence of a section  $s$  over  $M(x_0, C_{i'j'})$ , i. e., for points  $(\langle f \rangle, \langle p \rangle)$  such that  $p_0 \neq 0$  and  $C_{i'j'} \neq 0$ . Then one can put  $\langle p \rangle = \langle 1, \xi, \eta \rangle$  and  $C_{i'j'} = 1$ , and then from the condition  $f(p) = 0$  one concludes that

$$f(x_0, x_1, x_2) = (x_1 - \xi x_0)G(x_0, x_1, x_2, \{C_{ij}\}_{(i,j) \neq (i',j')}) + (x_2 - \eta x_0)H(x_0, x_1, x_2, \{C_{ij}\}_{(i,j) \neq (i',j')}),$$

hence one can define a section of  $\nu$  over  $M(x_0, C_{i'j'})$  by the rule

$$(\langle f \rangle, \langle 1, \xi, \eta \rangle) \mapsto \begin{pmatrix} x_1 - \xi x_0 & -H \\ x_2 - \eta x_0 & G \end{pmatrix}.$$

This proves the required statement.  $\square$

Using the existence of a local section and Zariski main theorem one shows that  $X$  is a principal  $\mathbb{P}G$ -bundle over  $M$ . Hence  $M$  is a geometrical quotient.

**Lemma 2.6.** *Every morphism of two sheaves with resolution of the type (1) can be uniquely lifted to a morphism of resolutions.*

*Proof.* Follows from  $\text{Ext}^1(\mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2}, 2\mathcal{O}_{\mathbb{P}_2}(-d+1)) = \text{Hom}(\mathcal{O}_{\mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}_2}, 2\mathcal{O}_{\mathbb{P}_2}(-d+1)) = 0$ .  $\square$

This lemma implies the following.

**Proposition 2.7.** *The points of  $M$  are in one-to-one correspondence with the isomorphism classes of sheaves that possess resolutions of the type (1).*

### 3. STABILITY. UNIVERSAL CURVE AS A SUBVARIETY OF SIMPSON MODULI SPACE

**Lemma 3.1.** *Let  $C$  be a plane projective curve of degree  $d$  and let  $p$  be a point at  $C$ . Then the ideal sheaf of a point  $p$  at  $C$ , i. e., the sheaf  $\mathcal{I}$  given by the exact sequence*

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_C \rightarrow \mathbb{k}_p \rightarrow 0$$

*is stable.*

*Proof.* Since  $\mathcal{O}_C$  does not have zero dimensional torsion, the same is true for its subsheaf  $\mathcal{I}$ , i. e.,  $\mathcal{I}$  is pure-dimensional. Let  $\mathcal{E}$  be a proper subsheaf of  $\mathcal{I}$ . Since  $\mathcal{E}$  is one-dimensional, its Hilbert polynomial is  $am + b$ . If the multiplicity  $a$  equals  $d$ , then  $\mathcal{I}/\mathcal{E}$  is zero dimensional, hence  $p_{\mathcal{I}}(m) - p(\mathcal{E}) = h^0(\mathcal{I}/\mathcal{E}) > 0$ .

Assume  $a < d$ . Then using [8, Lemma 6.7], we obtain a curve  $S \subseteq C$  of degree  $s < d$  such that its ideal sheaf  $\mathcal{I}_S \subseteq \mathcal{O}_C$  contains  $\mathcal{E}$  and  $\mathcal{Q} := \mathcal{I}_S/\mathcal{E}$  is a zero-dimensional sheaf. Hence the Hilbert polynomial of  $\mathcal{E}$  is

$$\begin{aligned} P_{\mathcal{E}}(m) &= P_{\mathcal{I}_S}(m) - h^0(\mathcal{Q}) = P_{\mathcal{O}_C}(m) - P_{\mathcal{O}_S}(m) - h^0(\mathcal{Q}) = \\ &= dm + \frac{d(3-d)}{2} - (sm + \frac{s(3-s)}{2}) - h^0(\mathcal{Q}) \end{aligned}$$

Therefore,

$$p_{\mathcal{E}}(m) = m + \frac{3}{2} - \frac{d+s}{2} - \frac{h^0(\mathcal{Q})}{d-s}.$$

Since

$$p_{\mathcal{I}}(m) = m + \frac{(3-d)}{2} - \frac{1}{d},$$

one sees that  $p_{\mathcal{E}}(m) < p_{\mathcal{I}}(m)$  if and only if  $\frac{1}{d} < \frac{s}{2} + \frac{h^0(\mathcal{Q})}{d-s}$  or equivalently  $1 < \frac{sd}{2} + d \cdot \frac{h^0(\mathcal{Q})}{d-s}$ , which is clearly true since  $d \geq 3$ .

We proved  $p_{\mathcal{E}}(m) < p_{\mathcal{I}}(m)$  for every proper subsheaf  $\mathcal{E}$  of  $\mathcal{I}$ . Therefore,  $\mathcal{I}$  is stable.  $\square$

**Proposition 3.2.** 1) *The sheaves with resolution (1) are stable.*

2) *The corresponding map*

$$M \rightarrow M_{dm + \frac{d(3-d)}{2} + 1}(\mathbb{P}_2), \quad [\mathcal{F}] \mapsto [\mathcal{F}],$$

*is a closed embedding of codimension  $\frac{d(d-3)}{2}$ .*

*Proof.* 1) The isomorphism class of every sheaf  $\mathcal{F}$  with resolution (1) is represented by a plane projective curve  $C$  of degree  $d$  and a point  $p$  at  $C$ . One can see that  $\mathcal{F}$  is a non-trivial extension

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{F} \rightarrow \mathbb{k}_p \rightarrow 0$$

and can be obtained as  $\mathcal{E}xt^1(\mathcal{I}, \mathcal{O}_{\mathbb{P}_2})(-d)$ , where  $\mathcal{I}$  is the ideal sheaf of a point  $p$  at  $C$ , i. e.,  $\mathcal{I}$  is given by the exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_C \rightarrow \mathbb{k}_p \rightarrow 0.$$

Using the notation  $\mathcal{I}^D := \mathcal{E}xt^1(\mathcal{I}, \omega_{\mathbb{P}_2})$  from [9] we get  $\mathcal{F} = \mathcal{F}^D(-d+3)$ . By Lemma 3.1  $\mathcal{I}$  is stable. Therefore, by the result from [9] its dual  $\mathcal{I}^D$  is stable as well. Note that by [8, Lemma 9.2] it can not be properly semi-stable.

This proves the first part of the statement.

2) There is a family of sheaves, flat over  $X$ , with Hilbert polynomial  $dm + \frac{d(3-d)}{2} + 1$  given by the resolution

$$0 \rightarrow 2\mathcal{O}_{X \times \mathbb{P}_2}(-d+1) \xrightarrow{\Psi} \mathcal{O}_{X \times \mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{X \times \mathbb{P}_2} \rightarrow \mathcal{F} \rightarrow 0,$$

where  $\Psi|_{\{A\} \times \mathbb{P}_2} = A$ . Since  $X \xrightarrow{\nu} M$  is a  $\mathbb{P}G$ -bundle over  $M$ , we get locally over  $M$  a flat family of sheaves, which induces the required morphism.

Clearly the morphism is injective with a closed image. It remains to show that it is a closed embedding. In other words we need to consider its image equipped with the induced structure and show that the inverse map is a morphism. This can be done using the method from [7, 6.5]. Namely, given a point in  $M_{dm + \frac{d(3-d)}{2} + 1}(\mathbb{P}_2)$  represented by a sheaf  $\mathcal{F}$  with resolution (1), it is enough to construct a point of the universal curve of degree  $d$  from the Beilinson spectral sequence converging to  $\mathcal{F}$  (cf. [11, 3.1.4. Theorem II, page 245] and also [1]) by means of algebraic operations.

Since  $\mathcal{F}$  is a one-dimensional sheaf, the only non-trivial part of the first sheet of the Beilinson spectral sequence

$$E_1^{p,q}(\mathcal{F}) = H^q(\mathbb{P}_2, \mathcal{F} \otimes \Omega^{-p}(-p)) \otimes \mathcal{O}(p)$$

is a  $2 \times 3$  rectangular

$$E_1^{-2,1} \longrightarrow E_1^{-1,1} \longrightarrow E_1^{0,1}$$

$$E_1^{-2,0} \longrightarrow E_1^{-1,0} \longrightarrow E_1^{0,0}.$$

Analyzing this spectral sequence as in [2, 2.2] and taking into account the stability of  $\mathcal{F}$  one can conclude that  $\mathcal{F}$  is a non-trivial extension

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{F} \rightarrow \mathbb{k}_p \rightarrow 0,$$

where  $(C, p) \in M$  and the sheaves  $\mathcal{O}_C, \mathbb{k}_p$  can be computed in terms of cokernels of the maps involved in the Beilinson spectral sequence.  $\square$

**Remark 3.3.** *Note that the points of  $M$  can be seen as isomorphism classes of non-trivial extensions*

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{F} \rightarrow \mathbb{k}_p \rightarrow 0,$$

where  $(C, p) \in M$ .

#### 4. UNIVERSAL SINGULAR LOCUS AS THE SUBVARIETY OF SINGULAR SHEAVES

Let  $X'$  be the subvariety of matrices in  $X$  defining singular sheaves, i.e., sheaves that are not locally free on their support.

A matrix  $A \in X$  as in (1) defines a singular sheaf if and only if it vanishes at some point  $q$  of  $\mathbb{P}_2$ . Since the linear forms  $z_1$  and  $z_2$  are linear independent, this point could only be the common zero point of  $z_1$  and  $z_2$ . If  $z_1 = a_0x_0 + a_1x_1 + a_2x_2$  and  $z_2 = b_0x_0 + b_1x_1 + b_2x_2$ , then  $q = \langle d_0, d_1, d_2 \rangle$ , where  $d_i$  are the minors of the matrix  $\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$ . Hence  $X'$  is given as a closed subvariety in  $X$  by the equations

$$(2) \quad f_1 = q_1(d_0, d_1, d_2) = 0, \quad f_2 = q_2(d_0, d_1, d_2) = 0.$$

After computing the partial derivatives of  $f_1$  and  $f_2$  and taking into account that the minors  $d_0, d_1$ , and  $d_2$  do not vanish simultaneously since  $z_1$  and  $z_2$  are always linear independent, we conclude that  $X'$  is a smooth subvariety of codimension 2 in  $X$ .

**Lemma 4.1.** *A point  $(C, p)$  from  $M$  corresponds to a singular sheaf if and only if  $p$  is a singular point of  $C$ , i.e., the subvariety  $M'$  of singular sheaves coincides with the universal singular locus  $\{(C, p) \mid p \in \text{Sing}(C)\}$ .*

*Proof.* Let  $(C, p)$  be a point in  $M$ . Then there is a matrix  $A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} \in X$  such that  $C$  is the zero set of  $f = \det A$  and  $p$  is the common zero set of  $z_1$  and  $z_2$ .

Suppose  $(C, p)$  corresponds to a singular sheaf. Then  $q_1(p) = q_2(p) = 0$  and one checks that  $\frac{\partial f}{\partial x_i}(p) = 0$  for all  $i = 0, 1, 2$ .

If  $p$  is a singular point of  $C$ , then all partial derivatives  $\frac{\partial f}{\partial x_i}(p)$  vanish. Since  $\frac{\partial f}{\partial x_i}(p) = (\frac{\partial z_1}{\partial x_i}q_2 - \frac{\partial z_2}{\partial x_i}q_1)(p)$  and since  $z_1$  and  $z_2$  are linear independent, one concludes that  $q_1(p) = q_2(p) = 0$ , hence  $(C, p)$  defines a singular sheaf.  $\square$

Since  $X$  is a principal bundle over  $M$  and since  $X'$  is smooth, one concludes that  $M'$  is smooth as well. One can also show this directly. The codimension of  $M'$  in  $M$  is 2.

Let  $M_B = M \setminus M'$ , its points are isomorphism classes of vector bundles (on support). Then one could consider  $M$  as a compactification of  $M_B$  by coherent sheaves.

## 5. BLOW UP $\text{Bl}_{M'}(M)$ AS A COMPACTIFICATION OF $M_B$ BY VECTOR BUNDLES

For a fixed point  $(C, p) \in M'$  representing an isomorphism class  $[\mathcal{F}]$  of a singular sheaf and for a fixed tangent vector  $v \in T_{[\mathcal{F}]}M \setminus T_{[\mathcal{F}]}M'$ , i. e.,  $v$  is normal to  $M'$ , we are going to construct a 1-dimensional sheaf on the surface  $D(p)$  locally free on its support. We call such sheaves  $R$ -bundles. We are going to show that  $\mathbb{P}(T_{[\mathcal{F}]}M/T_{\mathcal{F}}M')$  is naturally the space of equivalence classes of  $R$ -bundles. We shall use the parameter space  $X$ .

Let  $A \in X'$  and  $B \in T_A X$  represent a singular sheaf  $[\mathcal{F}] = (C, p) \in M'$  and a tangent vector at  $[\mathcal{F}]$  respectively. Then pulling the morphism

$$2\mathcal{O}_{\mathbb{k} \times \mathbb{P}_2}(-d+1) \xrightarrow{A+tB} \mathcal{O}_{\mathbb{k} \times \mathbb{P}_2}(-d+2) \oplus \mathcal{O}_{\mathbb{k} \times \mathbb{P}_2},$$

back to the blowing up  $Z = \text{Bl}_{0 \times p}(\mathbb{k} \times \mathbb{P}_2)$ , factoring out the canonical section of the canonical divisor, and restricting the cokernel to the zero fibre of  $Z \rightarrow \mathbb{k}$  we obtain one-dimensional sheaves on  $D(p)$  given by locally free resolutions

$$(3) \quad 0 \rightarrow 2\mathcal{O}_{D(p)}(-d+2, -1) \xrightarrow{\Phi(A,B)} \mathcal{O}_{D(p)}(-d+2, 0) \oplus \mathcal{O}_{D(p)} \rightarrow \mathcal{E}(A, B) \rightarrow 0.$$

See [4, Section 4] and [5] for more details.

Assume without loss of generality that

$$(4) \quad p = \langle 1, 0, 0 \rangle, \quad A = \begin{pmatrix} x_1 & q_1 \\ x_2 & q_2 \end{pmatrix} \in X'.$$

We can write  $A$  as

$$\begin{pmatrix} x_1 & x_1(\sum_{i \geq 1} A_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j) + x_2(\sum_{i=0} A_{0j} x_0^{d-1-j} x_2^{j-1}) \\ x_2 & x_1(\sum_{i \geq 1} B_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j) + x_2(\sum_{i=0} B_{0j} x_0^{d-1-j} x_2^{j-1}) \end{pmatrix}, \quad A_{ij}, B_{ij} \in \mathbb{k}.$$

Straightforward calculations using (2) show that the tangent equation at  $A$  in this case are

$$(5) \quad \begin{cases} \xi_{00} = A_{10}\xi_0 + A_{01}\eta_0 \\ \eta_{00} = B_{10}\xi_0 + B_{01}\eta_0 \end{cases},$$

where

$$B = \begin{pmatrix} \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi_{00} x_0^{d-1} + \cdots + \xi_{0d-1} x_2^{d-1} \\ \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 & \eta_{00} x_0^{d-1} + \cdots + \eta_{0d-1} x_2^{d-1} \end{pmatrix}$$

is a tangent vector at  $A$ .

Then one computes

$$\Phi(A, B) = \begin{pmatrix} u_1 & u_1(\sum_{i \geq 1} A_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j) + u_2(\sum_{i=0} A_{0j} x_0^{d-1-j} x_2^{j-1}) \\ u_2 & u_1(\sum_{i \geq 1} B_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j) + u_2(\sum_{i=0} B_{0j} x_0^{d-1-j} x_2^{j-1}) \end{pmatrix} + \begin{pmatrix} \xi_0 & \xi_{00} x_0^{d-2} \\ \eta_0 & \eta_{00} x_0^{d-2} \end{pmatrix} u_0.$$

The cokernel of such a matrix is not a locally free sheaf on its support (defined by the determinant of the matrix) if and only if all entries of the matrix vanish at some point. Since by the construction this can only happen on  $D_1(p)$ , one sees that this condition is equivalent to

$$\xi_{00} = A_{10}\xi_0 + A_{01}\eta_0, \quad \eta_{00} = B_{10}\xi_0 + B_{01}\eta_0.$$

The latter are just the tangent equations of  $X'$  at  $A$ .

So, for every  $A \in X'$  and for every  $B \in T_A X \setminus T_A X'$  we obtain a sheaf  $\mathcal{E} = \mathcal{E}(A, B)$  on  $D(p)$  locally free on its support. Here  $p$  is the common zero of  $z_1$  and  $z_2$ . We will call such sheaves  **$R$ -bundles**.

**Lemma 5.1.** *Every morphism of two sheaves with resolution of the type (3) can be uniquely lifted to a morphism of resolutions.*

*Proof.* Follows from  $\text{Ext}^1(\mathcal{O}_{D(p)}(-d+2, 0) \oplus \mathcal{O}_{D(p)}, 2\mathcal{O}_{D(p)}(-d+2, -1)) = \text{Hom}(\mathcal{O}_{D(p)}(-d+2, 0) \oplus \mathcal{O}_{D(p)}, 2\mathcal{O}_{D(p)}(-d+2, -1)) = 0$ .

We are going to prove that the groups  $H^0(D(p), \mathcal{O}_{D(p)}(-d+2, -1))$ ,  $H^1(D(p), \mathcal{O}_{D(p)}(-d+2, -1))$ ,  $H^0(D(p), \mathcal{O}_{D(p)}(0, -1))$ , and  $H^1(D(p), \mathcal{O}_{D(p)}(0, -1))$  are zero.

Let us compute the cohomology groups of the sheaf  $\mathcal{O}_{D(p)}(-d+2, -1)$ . Consider the gluing exact sequence

$$0 \rightarrow \mathcal{O}_{D(p)}(-d+2, -1) \rightarrow \mathcal{O}_{D_0(p)}(-d+2, -1) \oplus \mathcal{O}_{D_1(p)}(-1) \rightarrow \mathcal{O}_L(-1) \rightarrow 0.$$

Since all the cohomology groups of  $\mathcal{O}_{D_1}(-L)$  and  $\mathcal{O}_L(-1)$  are zero, using the long exact cohomology sequence we conclude that  $H^i(D(p), \mathcal{O}_{D(p)}(-d+2, -1)) \cong H^i(D_0(p), \mathcal{O}_{D_0(p)}(-d+2, -1))$ . From the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_1}(-d+1, -2) \rightarrow \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_1}(-d+2, -1) \rightarrow \mathcal{O}_{D_0(p)}(-d+2, -1) \rightarrow 0$$

and the corresponding long exact cohomology sequence using that

$$H^0(\mathbb{P}_2 \times \mathbb{P}_1, \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_1}(-d+2, -1)) = 0, \quad H^1(\mathbb{P}_2 \times \mathbb{P}_1, \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_1}(-d+1, -2)) = 0$$

we conclude that  $H^0(D_0(p), \mathcal{O}_{D_0(p)}(-d+2, -1)) = 0$ . Using that

$$H^1(\mathbb{P}_2 \times \mathbb{P}_1, \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_1}(-d+2, -1)) = 0, \quad H^2(\mathbb{P}_2 \times \mathbb{P}_1, \mathcal{O}_{\mathbb{P}_2 \times \mathbb{P}_1}(-d+1, -2)) = 0,$$

we conclude that  $H^1(D_0(p), \mathcal{O}_{D_0(p)}(-d+2, -1)) = 0$ .

Analogously one computes that the cohomology groups of  $\mathcal{O}_{D(p)}(0, -1)$  are zero as well.  $\square$

**Remark 5.2.** *Note that the uniqueness of the lifting implies that the lifting of an isomorphism of  $R$ -bundles is an isomorphism in each degree.*

**Definition 5.3.** *Let  $\mathcal{E}_1 = \mathcal{E}(A, B_1)$  and  $\mathcal{E}_2 = \mathcal{E}(A, B_2)$  be two  $R$ -bundles on  $D(p)$ . We call them equivalent if there exists an automorphism  $\phi$  of  $D(p)$  that acts identically on  $D_0(p)$  and such that  $\phi^*(\mathcal{E}_1) \cong \mathcal{E}_2$ .*

**Proposition 5.4.** *Two  $R$ -bundles  $\mathcal{E}_1 = \mathcal{E}(A, B_1)$  and  $\mathcal{E}_2 = \mathcal{E}(A, B_2)$  are equivalent if and only if  $B_1$  and  $B_2$  represent the same point in  $\mathbb{P}N_A$ , where  $N = T_A X / T_A X'$ .*

*Proof.* “ $\Rightarrow$ ”. Let  $\mathcal{E}_1 = \mathcal{E}(A, B_1)$  and  $\mathcal{E}_2 = \mathcal{E}(A, B_2)$  be two equivalent  $R$ -bundles, then the sheaves  $\mathcal{E}_1$  and  $\mathcal{E}_2$  possess locally free resolutions of type (3), they are cokernels of  $\Phi_1 = \Phi(A, B_1)$  and  $\Phi_2 = \Phi(A, B_2)$  respectively.

Equivalence of  $\mathcal{E}_1$  and  $\mathcal{E}_2$  means that there exists an isomorphism  $\phi : D(p) \rightarrow D(p)$  identical on  $D_0(p)$  such that there is an isomorphism  $\mathcal{E}_2 \xrightarrow{\xi} \phi^*(\mathcal{E}_1)$ . By Lemma 5.1  $\xi$  can be uniquely lifted to a morphism of resolutions

$$(6) \quad \begin{array}{ccccccc} 0 \rightarrow 2\mathcal{O}_{D(p)}(-d+2, -1) & \xrightarrow{\Phi_2} & \mathcal{O}_{D(p)}(-d+2, 0) \oplus \mathcal{O}_{D(p)} & \longrightarrow & \mathcal{E}_2 & \longrightarrow & 0 \\ & \downarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} & & & \downarrow \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{pmatrix} & & \downarrow \xi \\ 0 \rightarrow 2\mathcal{O}_{D(p)}(-d+2, -1) & \xrightarrow{\phi^*(\Phi_1)} & \mathcal{O}_{D(p)}(-d+2, 0) \oplus \mathcal{O}_{D(p)} & \rightarrow & \phi^*(\mathcal{E}_1) & \rightarrow & 0. \end{array}$$

Note that from the uniqueness of the lifting it follows that both matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $\begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{pmatrix}$  are invertible.

We are going to show now that for some  $\mu \in \mathbb{k}^*$  the matrix  $B_2 - \mu B_1$  satisfies the tangent equations (5), i. e.,  $B_2 - \mu B_1 \in T_A(X_8)$ . So  $B_1$  and  $B_2$  represent the same element in  $\mathbb{P}N_A$ . Let us present here a detailed proof.

One can assume without loss of generality that  $A$  is as in (4). Let

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mathbb{P}_2 \rightarrow \mathbb{P}_2, \quad \langle u_0, u_1, u_2 \rangle \mapsto \langle (u_0, u_1, u_2) \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle$$

be the restriction of  $\phi$  to  $D_0(p)$ . Let us consider the equality  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \phi^*(\Phi_1) = \Phi_2 \cdot \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{pmatrix}$ .

For the entry 1.1 this gives us the equality

$$a(u_1 + \beta u_0) + b(u_2 + \gamma u_0) + (a\xi_0 + b\eta_0)\alpha u_0 = \bar{a}u_1 + \bar{a}\mu_0 u_0$$

and hence the comparison of the coefficients yields

$$(7) \quad a = \bar{a}, \quad b = 0, \quad \beta + \xi_0\alpha = \mu_0.$$

For the entry 2.1 this gives us the equality

$$c(u_1 + \beta u_0) + d(u_2 + \gamma u_0) + (c\xi_0 + d\eta_0)\alpha u_0 = \bar{a}u_2 + \bar{a}\nu_0 u_0$$

and hence

$$(8) \quad c = 0, \quad d = \bar{a}, \quad \gamma + \eta_0\alpha = \nu_0.$$

For the entry 1.2 this gives the equality

$$\begin{aligned} a(u_1 + \beta u_0) \left( \sum_{i \geq 1} A_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j \right) + a(u_2 + \gamma u_0) \left( \sum_{i=0} A_{0j} x_0^{d-1-j} x_2^{j-1} \right) + a\xi_{00}\alpha x_0^{d-2} u_0 = \\ \bar{b}u_1 + \bar{d}u_1 \left( \sum_{i \geq 1} A_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j \right) + \bar{d}u_2 \left( \sum_{i=0} A_{0j} x_0^{d-1-j} x_2^{j-1} \right) + \bar{b}\mu_0 u_0 + \bar{d}\mu_{00} x_0^{d-2} u_0. \end{aligned}$$

Restricting this to  $D_1(p)$  gives

$$\begin{aligned} a(u_1 + \beta u_0)(A_{10}x_0^{d-2}) + a(u_2 + \gamma u_0)(A_{01}x_0^{d-2}) + a\xi_{00}\alpha x_0^{d-2} u_0 = \\ \bar{b}_{00}x_0^{d-2} u_1 + \bar{d}u_1(A_{10}x_0^{d-2}) + \bar{d}u_2(A_{01}x_0^{d-2}) + (\bar{b}_{00}\mu_0 + \bar{d}\mu_{00})x_0^{d-2} u_0 \end{aligned}$$

and hence

$$(9) \quad aA_{10} = \bar{b}_{00} + \bar{d}A_{10}, \quad aA_{01} = \bar{d}A_{01}, \quad a\beta A_{10} + a\gamma A_{01} + a\alpha\xi_{00} = \mu_0\bar{b}_{00} + \bar{d}\mu_{00}.$$

For the entry 2.2 this gives the equality

$$\begin{aligned} a(u_1 + \beta u_0) \left( \sum_{i \geq 1} B_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j \right) + a(u_2 + \gamma u_0) \left( \sum_{i=0} B_{0j} x_0^{d-1-j} x_2^{j-1} \right) + a\xi_{00}\alpha x_0^{d-2} u_0 = \\ \bar{b}u_2 + \bar{d}u_1 \left( \sum_{i \geq 1} B_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j \right) + \bar{d}u_2 \left( \sum_{i=0} B_{0j} x_0^{d-1-j} x_2^{j-1} \right) + \bar{b}\nu_0 u_0 + \bar{d}\nu_{00} x_0^{d-2} u_0. \end{aligned}$$



Restricting this to  $D_1(p)$  gives

$$a(u_1 + \beta u_0)(B_{10}x_0^{d-2}) + a(u_2 + \gamma u_0)(B_{01}x_0^{d-2}) + a\eta_{00}\alpha x_0^{d-2}u_0 = \\ \bar{b}_{00}x_0^{d-2}u_2 + \bar{d}u_1(B_{10}x_0^{d-2}) + \bar{d}u_2(B_{01}x_0^{d-2}) + (\bar{b}_{00}\nu_0 + \bar{d}\nu_{00})x_0^{d-2}u_0$$

and hence

$$(10) \quad aB_{10} = \bar{d}B_{10}, \quad aB_{01} = \bar{b}_{00} + \bar{d}B_{01}, \quad a\beta B_{10} + a\gamma B_{01} + a\alpha\eta_{00} = \nu_0\bar{b}_{00} + \bar{d}\nu_{00}.$$

From (7) and (8) one obtains  $\beta = \mu_0 - \alpha\xi_0$  and  $\gamma = \nu_0 - \alpha\eta_0$ . Then using (9) we get

$$\begin{aligned} \bar{d}\mu_{00} - a\alpha\xi_{00} &= a\beta A_{10} + a\gamma A_{01} - \mu_0\bar{b}_{00} = \\ &= a(\mu_0 - \alpha\xi_0)A_{10} + a(\nu_0 - \alpha\eta_0)A_{01} - \mu_0A_{10}(a - \bar{d}) = \\ &= A_{10}(\bar{d}\mu_0 - a\alpha\xi_0) + aA_{01}\nu_0 - a\alpha\eta_0A_{01} = \\ &= A_{10}(\bar{d}\mu_0 - a\alpha\xi_0) + \bar{d}A_{01}\nu_0 - a\alpha\eta_0A_{01} = \\ &= A_{10}(\bar{d}\mu_0 - a\alpha\xi_0) + A_{01}(\bar{d}\nu_0 - a\alpha\eta_0). \end{aligned}$$

And using (10) we get

$$\begin{aligned} \bar{d}\nu_{00} - a\alpha\eta_{00} &= a\beta B_{10} + a\gamma B_{01} - \nu_0\bar{b}_{00} = \\ &= a(\mu_0 - \alpha\xi_0)B_{10} + a(\nu_0 - \alpha\eta_0)B_{01} - \nu_0B_{01}(a - \bar{d}) = \\ &= B_{10}(\bar{d}\mu_0 - a\alpha\xi_0) + B_{01}(\bar{d}\nu_0 - a\alpha\eta_0). \end{aligned}$$

Therefore,  $\bar{d}B_1 - a\alpha B_2$  satisfies (5), hence  $B_1 - (\bar{d}^{-1}a\alpha) \cdot B_2 \in T_A X'$ . This means that  $B_1$  and  $B_2$  define the same point in  $\mathbb{P}N_A$ .

“ $\Leftarrow$ ”. Let now  $B_1$  and  $B_2$  be two equivalent normal vectors at  $A \in X'$ . Let  $\Phi_1 = \Phi(A, B_1)$  and  $\Phi_2 = \Phi(A, B_2)$  be the matrices defining as in (3) the sheaves  $\mathcal{E}_1$  and  $\mathcal{E}_2$  respectively. Since  $B_1$  and  $B_2$  define the same point in  $\mathbb{P}N_A$ , it follows that

$$B_2 - \alpha \cdot B_1 \in T_A(X_8)$$

for some  $\alpha \in \mathbb{k}^*$ .

Let

$$B_1 = \begin{pmatrix} \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 & \xi_{00} x_0^{d-1} + \cdots + \xi_{0d-1} x_2^{d-1} \\ \eta_0 x_0 + \eta_1 x_1 + \eta_2 x_2 & \eta_{00} x_0^{d-1} + \cdots + \eta_{0d-1} x_2^{d-1} \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} \mu_0 x_0 + \mu_1 x_1 + \mu_2 x_2 & \mu_{00} x_0^{d-1} + \cdots + \mu_{0d-1} x_2^{d-1} \\ \nu_0 x_0 + \nu_1 x_1 + \nu_2 x_2 & \nu_{00} x_0^{d-1} + \cdots + \nu_{0d-1} x_2^{d-1} \end{pmatrix}.$$

Take

$$\beta = \mu_0 - \xi_0 \alpha, \quad \gamma = \nu_0 - \eta_0 \alpha,$$

and let

$$(11) \quad \phi_1 = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mathbb{P}_2 \rightarrow \mathbb{P}_2, \quad \langle u_0, u_1, u_2 \rangle \mapsto \langle (u_0, u_1, u_2) \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle.$$

Note that the automorphisms of the form (11) are exactly the automorphisms of  $D_1 \cong \mathbb{P}_2$  acting identically on  $L$ .

Consider  $\phi : D(p) \rightarrow D(p)$  such that  $\phi|_{D_1} = \phi_1$  and  $\phi|_{D_0} = \text{id}_{D_0}$ . Using the tangent equations (5) and that  $u_0x_1 = u_0x_2 = 0$  one checks that  $\phi^*(\Phi_1) = \Phi_2$ . Indeed,

$$\begin{aligned} \phi^*(\Phi_1) &= \begin{pmatrix} u_1 + \beta u_0 & (u_1 + \beta u_0)(\sum_{i \geq 1} A_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j) + (u_2 + \gamma u_0)(\sum_{i=0} A_{0j} x_0^{d-1-j} x_2^{j-1}) \\ u_2 + \gamma u_0 & (u_1 + \beta u_0)(\sum_{i \geq 1} B_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j) + (u_2 + \gamma u_0)(\sum_{i=0} B_{0j} x_0^{d-1-j} x_2^{j-1}) \end{pmatrix} + \\ &\begin{pmatrix} \xi_0 & \xi_{00} x_0^{d-2} \\ \eta_0 & \eta_{00} x_0^{d-2} \end{pmatrix} \alpha u_0 = \begin{pmatrix} u_1 & u_1(\sum_{i \geq 1} A_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j) + u_2(\sum_{i=0} A_{0j} x_0^{d-1-j} x_2^{j-1}) \\ u_2 & u_1(\sum_{i \geq 1} B_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j) + u_2(\sum_{i=0} B_{0j} x_0^{d-1-j} x_2^{j-1}) \end{pmatrix} + \\ &\begin{pmatrix} \beta + \xi_0 \alpha & (\xi_{00} \alpha + \beta A_{10} + \gamma A_{01}) x_0^{d-2} \\ \gamma + \eta_0 \alpha & (\eta_{00} \alpha + \beta B_{10} + \gamma B_{01}) x_0^{d-2} \end{pmatrix} u_0 = \\ &\begin{pmatrix} u_1 & u_1(\sum_{i \geq 1} A_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j) + u_2(\sum_{i=0} A_{0j} x_0^{d-1-j} x_2^{j-1}) \\ u_2 & u_1(\sum_{i \geq 1} B_{ij} x_0^{d-1-i-j} x_1^{i-1} x_2^j) + u_2(\sum_{i=0} B_{0j} x_0^{d-1-j} x_2^{j-1}) \end{pmatrix} + \begin{pmatrix} \mu_0 & \mu_{00} x_0^{d-2} \\ \nu_0 & \nu_{00} x_0^{d-2} \end{pmatrix} u_0 = \Phi_2. \end{aligned}$$

Therefore, there is an isomorphism  $\phi^*(\mathcal{E}_1) \cong \mathcal{E}_2$ , which means that the sheaves  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are equivalent.  $\square$

This proposition immediately implies the main statement of this note.

**Theorem 5.5.** *Let  $\widetilde{M} = \text{Bl}_{M'}(M)$ . Then the exceptional divisor of this blow up consists of the equivalence classes of  $R$ -bundles.*

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